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Stability Analysis of Some Fractional Differential Equations by Special type of Spline Function

Faraidun K. Hamasalh[1] & Amina H. Ali[1]

1 Faculty of Science and Science Education, School of Science Education, Sulaimani University, Sulaimani, Iraq
E-mail : faraidun.hamasalh@univsul.edu.iq

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Abstract

One fractional spline interpolation is presented in this paper for the solution fractional initial value problems that is with spline interpolation as a new class depended on class C^2 −splines as a method to approximate the exact solution of such problem. Error bounds were discussed as the presented spline function. On other hands, stability analysis has been accomplished and two examples were chosen as consideration for the numerical explanation of the presented technique. The outcome demonstrates that the presented fractional cubic spline function which interpolates the lacunary data acted as an efficient and effective way to solve such kind Special problems.

Introduction

Fractional calculus is a field of mathematics study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value.[13]

Spline functions are commonly used in several areas such as these examples (data fitting, numerical solution of integral equations Lacunary interpolation, Interpolation and numerical solution of the ordinary partial differential equation) (see [18] and [19]).

Initial value problems occur in many branches of sciences and engineering, for fluid dynamics, quantum mechanics, optimal problems, etc., for this reason, the numerical solution of this problem is very important because the analytic solution is not always possible.

Many authors have used several different kinds of Spline functions as a solution for initial values problems (see [1], [15], [16] and [17]) are as examples, but these ideas are used as benefit way in this paper to solve fractional initial value problems.

A new class of cubic C^2 −lacunary spline interpolation is made and applied to find a numerical solution of fractional initial value

$$y^{\left(\frac{3}{2}\right)}(x) = f(x,y), y(x_0) = y_0\,, y'(x_0) = {y_0}', x \in [0,1] \tag{1}$$

Construction of the Fractional Spline Function

Construction of the spline interpolation function $s(x) \in C^2[0,1]$ which satisfies (1) at the knots $s(x_i) = ih\,, i = 0,1,\dots,N$ and $h = \frac{1}{N}$ is the main aim of this section, and this developed by constructed fractional spline function as given in [1] and [8] respectively. The class C^2- spline function is defined as follows:

$S^2{}_{(n,\frac{7}{2})} = \{s(x); s \in C^2[0,1], s \in P_{(\frac{7}{2})}(x), x \in I = [x_i, x_{i+1}]$ where $P_{\frac{7}{2}}(x)$ is the set of all polynomials of degree at most $\frac{7}{2}$.

Definition: [10]

The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-\xi)^{n-\alpha-1} f(\xi)d\xi, \quad n-1 < \alpha < n, n \in \mathbb{N}$

Definition: [8,14]

The Caputo fractional derivative of order $\alpha > 0$, is defined by:

$$D_a^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(s)}{(x-s)^{\alpha+1-n}}ds & \text{for } n-1 < \alpha < n, n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x) & \text{for } \alpha = n, \quad n \in \mathbb{N} \end{cases}$$

Existences and Uniqueness

Theorem 1:

Let $f(x) \in C^2$ and given the real numbers $s_i^{\frac{3}{2}}$, $i = 0, 1, \dots, N$, s_0, $s_0^{\frac{1}{2}}$ and $s_0^{\frac{5}{2}}$, then there exist a unique spline $s(x) \in S^2{}_{(n,\frac{7}{2})}$ Such that

$$\left.\begin{array}{r} s_0 = f_0 \\ s_0^{(\frac{1}{2})} = f_0^{(\frac{1}{2})} \\ s_i^{(\frac{3}{2})} = f_i^{(\frac{3}{2})}, i = 0,1,\dots,N \\ \text{and} \\ s_0^{(\frac{5}{2})} = f_0^{(\frac{5}{2})} \end{array}\right\} \tag{2}$$

Proof:

The unique spline function $s(x) \in S^2{}_{(n,\frac{7}{2})}$ *in* $[x_i, x_{i+1}]$ where $\in (0,1]$, we developed the construction spline function from [1,15,16], as follows:

$s(x) = A(x)s_i + B(x)h^{\frac{1}{2}}s^{\frac{1}{2}}{}_i + h^{\frac{3}{2}}\left[C(x)s^{\frac{3}{2}}{}_i + D(x)s^{\frac{3}{2}}{}_{i+\beta}\right] + E(x)h^{\frac{5}{2}}s^{\frac{5}{2}}{}_i$

where

$$\begin{aligned} A(x) &= 1, \\ B(x) &= \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}, \\ C(x) &= \frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}} - \frac{32}{\beta^2 105\sqrt{\pi}}x^{\frac{7}{2}}, \\ D(x) &= \frac{32}{\beta^2 105\sqrt{\pi}}x^{\frac{7}{2}}, \\ E(x) &= \frac{8}{15\sqrt{\pi}}x^{\frac{5}{2}} - \frac{32}{\beta^2 105\sqrt{\pi}}x^{\frac{7}{2}}. \end{aligned} \tag{3}$$

and $= x_i + t\beta h$, $0 \le t \le 1$, with a similar expression for $s(x)$ *in* $[x_{i-1}, x_i]$. Since $s(x) \in C^2$ and $s(xi^+) = s(xi^-)$ to $s^{\frac{5}{2}}(xi^+) = s^{\frac{5}{2}}(xi^-)$

respectively, for *i*=0, 1, 2, …, *N*, leads to the following linear system of equations:

$$s_i = s_{i-1} + \frac{2}{\sqrt{\pi}}\beta^{\frac{1}{2}}h^{\frac{1}{2}}s^{\frac{1}{2}}{}_{i-1} + h^{\frac{3}{2}}\beta^{\frac{3}{2}}\left[\frac{108}{105\sqrt{\pi}}s^{\frac{3}{2}}{}_{i-1} + \frac{32}{105\sqrt{\pi}}s^{\frac{3}{2}}{}_{i-1+\beta}\right] + \frac{24}{105\sqrt{\pi}}\beta^{\frac{5}{2}}h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i-1}, \tag{4}$$

$$h^{\frac{1}{2}}s^{\frac{1}{2}}{}_{i} = h^{\frac{1}{2}}s^{\frac{1}{2}}{}_{i-1} + \frac{\beta}{3}h^{\frac{3}{2}}s^{\frac{3}{2}}{}_{i-1+\beta} + \frac{\beta^2}{6}h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i-1}, \tag{5}$$

$$s^{\frac{3}{2}}{}_{i} = f(x_i, s_i), s^{\frac{3}{2}}{}_{i} = s^{\frac{3}{2}}{}_{i-1}, \tag{6}$$

$$h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i} = \frac{2}{\beta}h^{\frac{3}{2}}s^{\frac{1}{2}}{}_{i-1+\beta} - \frac{2}{\beta}h^{\frac{3}{2}}s^{\frac{3}{2}}{}_{i-1} - h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i-1} \quad , i = 1, \dots, N, \tag{7}$$

And hence $s(x)$ is uniquely determined in [0,1].

Corollary:

Let $\beta = 1$ in the Theorem 1, then the following system is stable.

Proof:

Directly from the equation (4), (5) and (7) we get the following system

$$s_i = s_{i-1} + \frac{2}{\sqrt{\pi}}h^{\frac{1}{2}}s^{\frac{1}{2}}{}_{i-1} + h^{\frac{3}{2}}\left[\frac{108}{105\sqrt{\pi}}s^{\frac{3}{2}}{}_{i-1} + \frac{32}{105\sqrt{\pi}}s^{\frac{3}{2}}{}_{i}\right] + \frac{24}{105\sqrt{\pi}}h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i-1}, \tag{8}$$

$$h^{\frac{1}{2}}s^{\frac{1}{2}}{}_{i} = h^{\frac{1}{2}}s^{\frac{1}{2}}{}_{i-1} + \frac{1}{3}h^{\frac{3}{2}}s^{\frac{3}{2}}{}_{i} + \frac{1}{6}h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i-1}, \tag{9}$$

$$h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i} = 2h^{\frac{3}{2}}s^{\frac{1}{2}}{}_{i} - 2h^{\frac{3}{2}}s^{\frac{3}{2}}{}_{i-1} - h^{\frac{5}{2}}s^{\frac{5}{2}}{}_{i-1} \quad , i = 1, \dots, N. \tag{10}$$

And in the section of Stability analysis we will show that the above system will be stable.

Theorem 2: [2,3,20]

Let $g \in C^{2m}[0.h]$ be given, and let P_{2m-1} be the unique Hermite interpolation polynomial of degree $2m-1$ that matches g and its first $m-1$ derivatives $g^{(r)}$ at 0 and h. Then

$$\left|e^{(r)}(x)\right| \le \frac{h^r[x(h-x)]^{m-r}G}{r!(2m-1)!}, r = 0(1)m, 0 \le x \le h, \tag{11}$$

where

$\left|e^{(r)}(x)\right| = \left|g^{(r)}(x) - P^{(r)}{}_{2m-1}(x)\right|$ and

$$G = \max_{0\le x\le h} \max_{0\le x\le h}\left|g^{(2m)}(x)\right| \tag{12}$$

The bounds in (11) are best possible for $r = 0$ only.

Theorem 3:

Suppose that s(x) be the fractional spline defined in Theorem 2, $f^{(\frac{1}{2})}$, $f^{(\frac{3}{2})} \in C^2[0,1]$ and that $f^{(p)}(0) = 0, p = 1,2$ then for any $x \in [0,1]$ we have

$$|s(x) - f(x)| \le \frac{h^2}{2\sqrt{\pi}}f^{(\frac{5}{2})}$$

Proof:

Because $s^{(\frac{1}{2})}(x)$ is Hermite interpolation polynomial of degree 3 matching $f^{(\frac{1}{2})}(x)$, $f^{(\frac{3}{2})}(x)$ at $x = x_i, x_{i+1}$ so for any $x \in [x_i, x_{i+1}]$, we have using (11) with $m = 1, g = f^{(\frac{1}{2})}$ and $p_3 = s^{(\frac{3}{2})}(x)$, we get:

$|s^{(\frac{1}{2})} - f^{(\frac{1}{2})}| \le \frac{h^2}{1!(4)}D^{(2)}D^{(\frac{1}{2})}f$ also if we put $g = f^{(\frac{3}{2})}$ and $P_3 = s^{(\frac{3}{2})}$ we get $|s^{(\frac{3}{2})} - f^{(\frac{3}{2})}| \le \frac{h^2}{1!(4)}D^{(2)}D^{(\frac{3}{2})}f$

$|I^{\frac{1}{2}}{}_{0|x}\left[s^{\left(\frac{1}{2}\right)} - f^{\left(\frac{1}{2}\right)}\right]| \le I^{\frac{1}{2}}{}_{0|x}[\frac{h^2}{1!(4)}D^{(2)}D^{(\frac{1}{2})}f]$ then we can get

$$|s(x) - s(0) + f(0) - f(x)| \le \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}[\frac{h^2}{4}D^{(2)}D^{(\frac{1}{2})}f]$$

Since $s(0) = f(0)$ and $x \in [0,1]$ then the last equation becomes

$|s(x) - f(x)| \le \frac{h^2}{2\sqrt{\pi}}D^{(2)}D^{(\frac{1}{2})}f$ and since $f^{(p)}(0) = 0, p = 1, 2,$ [13] we have

$D^{(2)}D^{(\frac{1}{2})}f = D^{(\frac{5}{2})}f = f^{(\frac{5}{2})}$,

which implies that:

$$|s(x) - f(x)| \le \frac{h^2}{2\sqrt{\pi}}f^{(\frac{5}{2})} .$$

Stability Analysis

The presented methods (8), (9) and (10) are under consideration for testing its stability analysis and executing the method to the test the equation

$$y^{(\frac{3}{2})}(x) = -\lambda^{\frac{3}{2}} y(x), \qquad \lambda \in \mathbb{R}, y(x_0) = y_0, y'(x_0) = y', \tag{13}$$

Setting $h = K$, using equation (13) to obtain

$$s_i = \frac{105\sqrt{\pi}+108\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}} s_{i-1} + \frac{210h^{\frac{1}{2}}}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}} s^{\frac{1}{2}}{}_{i-1} + \frac{24h^{\frac{5}{2}}}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}} s^{\frac{5}{2}}{}_{i-1},$$

$$h^{\frac{1}{2}} s^{\frac{1}{2}}{}_i = \left[\frac{35\sqrt{\pi}+36\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}\right] s_{i-1} + \left[\frac{105\sqrt{\pi}+38\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}\right] h^{\frac{1}{2}} s^{\frac{1}{2}}{}_{i-1} + \left[\frac{105\sqrt{\pi}+16\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}{6(105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}})}\right] h^{\frac{5}{2}} s^{\frac{5}{2}}{}_{i-1},$$

$$h^{\frac{5}{2}} s^{\frac{5}{2}}{}_i = \left[\frac{152\lambda^3 h^3}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}\right] s_{i-1} + \left[\frac{40\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}\right] h^{\frac{1}{2}} s^{\frac{1}{2}}{}_{i-1} + \left[\frac{-105\sqrt{\pi}+80\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}{105\sqrt{\pi}-32\lambda^{\frac{3}{2}}h^{\frac{3}{2}}}\right] h^{\frac{5}{2}} s^{\frac{5}{2}}{}_{i-1}.$$

Or in matrix notation

$$S_i = MS_{i-1}, i = 1, \dots, \mathbb{N} \text{ where } \quad S_i = \begin{bmatrix} s_i \\ s^{\frac{1}{2}}{}_i \\ s^{\frac{5}{2}}{}_i \end{bmatrix}, \quad S_{i-1} = \begin{bmatrix} s_{i-1} \\ s^{\frac{1}{2}}{}_{i-1} \\ s^{\frac{5}{2}}{}_{i-1} \end{bmatrix} \quad \text{and}$$

$$M = \begin{bmatrix} \frac{105\sqrt{\pi}+108K^{\frac{3}{2}}}{105\sqrt{\pi}-32K^{\frac{3}{2}}} & \frac{210h^{\frac{1}{2}}}{105\sqrt{\pi}-32K^{\frac{3}{2}}} & \frac{24h^{\frac{5}{2}}}{105\sqrt{\pi}-32K^{\frac{3}{2}}} \\ \left[\frac{35\sqrt{\pi}+36K^{\frac{3}{2}}}{105\sqrt{\pi}-32K^{\frac{3}{2}}}\right] & \left[\frac{105\sqrt{\pi}+38K^{\frac{3}{2}}}{105\sqrt{\pi}-32K^{\frac{3}{2}}}\right] & \left[\frac{105\sqrt{\pi}+16K^{\frac{3}{2}}}{6(105\sqrt{\pi}-32K^{\frac{3}{2}})}\right] \\ \left[\frac{152K^3}{105\sqrt{\pi}-32K^{\frac{3}{2}}}\right] & \left[\frac{40K^{\frac{3}{2}}}{105\sqrt{\pi}-32K^{\frac{3}{2}}}\right] & \left[\frac{-105\sqrt{\pi}+80K^{\frac{3}{2}}}{105\sqrt{\pi}-32K^{\frac{3}{2}}}\right] \end{bmatrix}$$

The characteristic equation can be expressed as below:

$$r^3 - (trM)r^2 + (M_1 + M_2 + M_3)r - \det(M) = 0. \tag{14}$$

The cofactors of the diagonal elements can be denoted by M_1, M_2, M_3 respectively where r is the eigenvalue ,In addition, (8), (9) and (10) can define the fractional spline approximation method that have interval of periodicity$(0, K^{\frac{3}{2}})$, where the eigenvalues $r_{1,2}$ of the matrix M are complex conjugate and $|r_3| \leq 1$. If all complex eigenvalues have negative real parts, the characteristic equation will be stable as the characteristic polynomial (14) says. [1], [8] and [11].

Numerical Examples

The class C^2 −fractional interpolation spline to be explained and the applicability of the presented method to be showed computationally, two fractional initial value problems are considered. The effectiveness of the suggested technique can be demonstrated by the applicability of the outcome respect these two Table 1 and 2 are shown as follows.

The notation e, $e^{\frac{1}{2}}$ and $e^{\frac{3}{2}}$ stands for the maximum magnitude errors $|s(x) - f(x)|$, $|s^{(\frac{1}{2})} - f^{(\frac{1}{2})}|$ and $|s^{(\frac{3}{2})} - f^{(\frac{3}{2})}|$ respectively.

Example 1: [4]

Consider the following nonlinear FIVP

$$D^{\frac{3}{2}} y(x) + y^2(x) = \frac{\Gamma(6)}{\Gamma(4.5)} x^{\frac{7}{2}} - \frac{3\Gamma(5)}{\Gamma(3.5)} x^{\frac{5}{2}} + \frac{\Gamma(4)}{\Gamma(2.5)} x^{\frac{3}{2}} + [x^5 - 3x^4 + 2x^3]^2,$$

With initial condition $y(0) = 0, y'(0) = 0$ and the exact solution is $y(x) = x^5 - 3x^4 + 2x^3$.

Table -1: Absolute error of s(x) and its derivatives of Example 1.

h	Absolute Error			Exact solution	Approximation solution
	e	$e^{\frac{1}{2}}$	$e^{\frac{3}{2}}$		
0.02	1.5×10^{-4}	1.3×10^{-4}	8.03×10^{-4}	1.9701×10^{-6}	1.28×10^{-4}
0.05	0.00016	0.00014	0.00097	1.2042×10^{-4}	1.958×10^{-5}
0.1	0.0076	0.00067	0.0548	0.0013	6.3×10^{-4}

Example 2: [12]

Consider the following linear FIVP

$D^{\frac{3}{2}}y(x) = -y(x)$, with initial condition $y(0) = 1 - \Gamma\left(\frac{1}{2}\right), y'(0) = -\Gamma\left(\frac{1}{2}\right)$, the exact solution is

$$y = 1 + \Gamma\left(\frac{1}{2}\right) + x\Gamma\left(\frac{1}{2}\right) + \frac{1-\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}} + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}} + \frac{1+\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)}x^3 - \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(5)}x^4 - \frac{1+\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}x^{\frac{9}{2}}.$$

Table -2: Absolute error of s(x) and its derivatives of Example 2.

h	Absolute Error			Exact solution	Approximation solution
	e	$e^{\frac{1}{2}}$	$e^{\frac{3}{2}}$		
0.02	2.4×10^{-4}	2.08×10^{-4}	8.2×10^{-5}	2.7896	2.7893
0.05	0.0017	0.0076	4.2×10^{-4}	2.8389	2.8313
0.1	0.0076	0.0067	0.0014	2.9179	2.9112

Conclusion

The application of the spline function has been studied rarity. So, this led to use the fractional spline interpolation function as a proposal for finding fractional initial value problems. The above two examples that have presented showed that given algorithm can provide encouraging results and also yields a good approximation to the provided solution that a small step size h must be selected.

The solution of fractional initial value problems is not only an approximation that is targeted by this newly presented algorithm because it approximates the higher order derivatives as well. Also, a spline procedure is established based on a parameter $\beta \in [0,1]$, and $\beta = 1$is taken and work on stability on it.

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